Short Research Note No. 1-2013: Uncertainty relations of events:
Position and event density are complementary in case of normal distributions.

by Magnus Pirovino (magnus.pirovino@opiro.li) based also on discussions with Thomas Breuer (tb@fhv.at)

1. MOTIVATION AND PREVIOUS DISCUSSIONS

In our previous discussions we investigated a situation where a measurement device $M(\underline{\Delta x})$ covering a section $\underline{\Delta x}$ of the real axis can detect an event ⓔ. The location of this event ⓔ is then known with precision Δx. The event density of this event was given by $a(x) :=$ ⓔ$d(x)$, where $d(x)$ is the unknown density of our event on the real axis. In such a measurement situation we developed the intuition that the uncertainty of location and event density should obey an uncertainty relation as follows

(1) "$\boldsymbol{\Delta a} \cdot \boldsymbol{\Delta x} =$ ⓔ"

Several attempts to derive a rigorous version of this intuitive equality failed by the lack of compatibility with the statistical interpretation of quantum mechanics as described by –what we call – the "Stanford paper" about the uncertainty principle (plato.stanford.edu/entries/qt-uncertainty/).

1.1. Motivation to investigate uncertainty relations of normal distributions

Since our intuitive uncertainty relation (1) is an equality (reaching a minimum) this gives rise to look at the situations in quantum mechanics where uncertainty relation inequalities reach their minimum (and turn into equalities). These situations are given by coherent states $\psi_0(x)$ obeying

$$var_{\psi_0}(\hat{x})var_{\psi_0}(\hat{p}) = \frac{\hbar^2}{4}.$$

As we will see in the next section the simplest coherent states ψ_0 are represented by normal distributions

$$g(\sigma, x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} \text{ where } \psi_0(x) = \sqrt{g(\sigma, x)}.$$

Thus, we can hope to find a rigorous version of (1) by investigating uncertainty relations of normal distributions. The major result of this brief paper is: If an event (action) is normally distributed with unknown parameters μ and σ, i.e. with an unknown normal density function, than, in fact, position and event density of events are complementary and we cannot determine by one single measurement both, location and event density, precisely. Equivalently, one cannot determine by one single measurement both location and the risk parameter σ of the normal distribution to arbitrary precision. If we know the location $\boldsymbol{x}$ **with precision** $\boldsymbol{\sigma}$ then we know the inverse of the risk parameter $\boldsymbol{\sigma^{-1}}$ **maximally with precision** $\sqrt{\frac{\pi-2}{2}}\boldsymbol{\sigma^{-1}}$.

2. COMPARISON TO COHERENT GROUND STATES OF A QUANTUM HARMONIC OSCILLATOR.

Consider a quantum harmonic oscillator as described by the following Hamiltonian

$$H = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$$

with momentum operator $\hat{p} = -i\hbar\frac{\partial}{\partial x}$ and position operator $\hat{x} = x$. From the quantum mechanics of the harmonic oscillator we know that the eigenstates ψ_n of H satisfy the following time-independent Schroedinger equation:

$$(2) \qquad H\psi_n = E_n\psi_n \ \ with \ \ E_n = \left(n + \tfrac{1}{2}\right)\hbar\omega, \ \ n = 0,1,2 \ldots.$$

Let $\sigma := \sqrt{\frac{\hbar}{2m\omega}}$ and $g(\sigma, x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}$ the density function of a normal distribution $g \backsim N(0, \sigma^2)$ around mean zero.

Then the (zero-particle) ground state ψ_0 (i.e. we consider the case $n = 0$ in equation (2)) is defined as follows:

$$(3) \qquad \psi_0(x) = \sqrt{g(\sigma, x)} = \sqrt{\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}}.$$

I.e. the square ${\psi_0}^2(x)$ of the ground-state is a normal distribution with zero mean and variance σ^2.

From the literature (see e.g. en.wikipedia.org/wiki/Normal_distribution#Fourier_transform_and_characteristic_function) we know that the Fourier transform of a normal distribution with mean zero and variance σ^2 is also a normal distribution with zero mean and variance σ^{-2}. A similar argument leads to the formulation of the following lemma.

Lemma 1
The Fourier transform of the ground state $\psi_0(x)$ is given by

$$\varphi_0(p) = \mathcal{F}(\psi_0)(p) = \sqrt{g\left(\frac{\hbar}{2\sigma}, p\right)} = \sqrt{\frac{1}{\sqrt{2\pi}\frac{\hbar}{2\sigma}} e^{-\frac{p^2}{2(\frac{\hbar}{2\sigma})^2}}}.$$

I.e. the square $\varphi_0(p)^2$ of the Fourier transform of the ground-state ψ_0 is a normal distribution too, with zero mean and variance $(\frac{\hbar}{2\sigma})^2$.

Proof. $\mathcal{F}(\psi_0)(p) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\sqrt{\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{x^2}{2\sigma^2}}}\,e^{-i\frac{px}{\hbar}}dx = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{\sqrt{2\pi}\sigma}}e^{-\frac{x^2}{4\sigma^2}}e^{-i\frac{px}{\hbar}}dx =$

$$= \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{\sqrt{2\pi}\sigma}}e^{-\frac{1}{4}(\frac{x}{\sigma}+i\frac{2p}{\hbar}\sigma)^2}e^{-\frac{p^2\sigma^2}{\hbar^2}}dx = \frac{1}{\sqrt{2\pi\hbar}}\frac{1}{\sqrt{\sqrt{2\pi}\sigma}}e^{-\frac{p^2\sigma^2}{\hbar^2}}\int_{-\infty}^{\infty}e^{-\frac{y^2}{2(\sqrt{2}\sigma)^2}}dy,$$

where $y = x - i\frac{2p\sigma^2}{\hbar}$ and $dy = dx$. Since $\int_{-\infty}^{\infty}e^{-\frac{y^2}{2(\sqrt{2}\sigma)^2}}dy = \sqrt{2\pi}(\sqrt{2}\sigma)$ this yields

$$\mathcal{F}(\psi_0)(p) = \sqrt{\frac{1}{\sqrt{2\pi}\frac{\hbar}{2\sigma}}e^{-\frac{p^2}{2(\frac{\hbar}{2\sigma})^2}}} = \sqrt{g\left(\frac{\hbar}{2\sigma}, p\right)} \qquad ■$$

Theorem 1
The ground state ψ_0 is coherent, i.e.

$$var_{\psi_0}(\hat{x})var_{\psi_0}(\hat{p}) = \frac{\hbar^2}{4} \;\; with \;\; var_{\psi_0}(\hat{x}) = \sigma^2 \; and \; var_{\psi_0}(\hat{p}) = (\frac{\hbar}{2\sigma})^2.$$

Proof. *The result follows directly from equation* (3), Lemma 1, *the fact that $E_{\varphi_0}(p) = 0$ and $var_{\psi_0}(\hat{p}) = E_{\varphi_0}(p^2) = (\frac{\hbar}{2\sigma})^2$.* ■

Definition 1
The action density (here also called event density) $\hat{a}$ for actions $\hbar$ in the ground state ψ_0 representing a normal distribution ${\psi_0}^2 = g \backsim N(0, \sigma^2)$ is defined in p-coordinates of the Fourier transform as follows:

$$\hat{a}\,\mathcal{F}(\psi_0)(p) := |p|\mathcal{F}(\psi_0)(p) = |p|\varphi_0(p)$$

with the expectation value

$$E_{\psi_0}(\hat{a}) = \int_{-\infty}^{\infty}|p|\,\mathcal{F}(\psi_0)(p)^2dp = \int_{-\infty}^{\infty}|p|\;\varphi_0(p)^2dp.$$

Remark: *$\hat{a}$ is not an observable represented by a linear Hermitian operator, but can be viewed as the absolute value of an observable represented by a Hermitian operator. Here the action density can be viewed as the absolute value of the momentum of a actions $\hbar$ in the ground state ψ_0.*

Lemma 2

$$E_{\psi_0}(\hat{a}) = \hbar\psi_0(0)^2 = \hbar g(\sigma, 0) = \frac{\hbar}{\sqrt{2\pi}\sigma}$$

I.e. the expected value of the action density is $\hbar$ times the density at the expected value of the location x.

Proof. $E_{\psi_0}(\hat{a}) = \int_{-\infty}^{\infty}|p|\ \varphi_0(p)^2 dp = 2\int_0^{\infty} p\ \varphi_0(p)^2 dp = 2\int_0^{\infty} p\ g\left(\frac{\hbar}{2\sigma}, p\right) dp = 2\int_0^{\infty} p\ \frac{1}{\sqrt{2\pi}\frac{\hbar}{2\sigma}} e^{-\frac{p^2}{2(\frac{\hbar}{2\sigma})^2}} dp =$ $\frac{-\hbar}{\sqrt{2\pi}\sigma}\int_0^{\infty} \frac{-p}{(\frac{\hbar}{2\sigma})^2} e^{-\frac{p^2}{2(\frac{\hbar}{2\sigma})^2}} dp = -\frac{\hbar}{\sqrt{2\pi}\sigma} e^{-\frac{p^2}{2(\frac{\hbar}{2\sigma})^2}}\Bigg|_0^{\infty} = \frac{\hbar}{\sqrt{2\pi}\sigma}$. ■

Lemma 3

$$E_{\psi_0}(\hat{a}^2) = E_{\psi_0}(\hat{p}^2)$$

Proof. It follows from the definition of the action density (Definition 1):

$E_{\psi_0}(\hat{a}) = \int_{-\infty}^{\infty}|p|^2\,\mathcal{F}(\psi_0)(p)^2 dp = \int_{-\infty}^{\infty} p^2\ \varphi_0(p)^2 dp = E_{\psi_0}(\hat{p}^2)$. ■

Lemma 4

$$var_{\psi_0}(\hat{a}) = var_{\psi_0}(\hat{p}) - E_{\psi_0}(\hat{a})^2 = \frac{\hbar^2}{4\sigma^2}(1 - \frac{2}{\pi})$$

Proof. *Since* $E_{\psi_0}(\hat{p}) = 0$ *and, by* Lemma 3, $E_{\psi_0}(\hat{a}^2) = E_{\psi_0}(\hat{p}^2)$ *it follows that:* $var_{\psi_0}(\hat{a}) = E_{\psi_0}(\hat{a}^2) - E_{\psi_0}(\hat{a})^2 = E_{\psi_0}(\hat{p}^2) - E_{\psi_0}(\hat{a})^2 = var_{\psi_0}(\hat{p}) - E_{\psi_0}(\hat{a})^2$. *The result now follows from* Theorem 1 *and* Lemma 2. ■

Theorem 2

If actions ħ occur in a (coherent) ground state ψ_0, *where* ${\psi_0}^2 = g \backsim N(0, \sigma^2)$ *then*

$$var_{\psi_0}(\hat{x})var_{\psi_0}(\hat{a}) = \frac{\hbar^2}{4}\left(1 - \frac{2}{\pi}\right) > \frac{\hbar^2}{11}.$$

I.e. action density and position of normally distributed actions ħ are complementary.

Proof. *The result follows directly from and* Lemma 1 *and* Theorem 1 *implying* $var_{\psi_0}(\hat{x}) = \sigma^2$. ■

3. Applications and Suggestions for Further Discussions

The findings of Lemma 2, Lemma 4 and Theorem 2 can potentially have broader applications than just in quantum physics. E.g. in statistics:

Application in statistics

Let X be a normally distributed random variable with unknown density $g(\mu, \sigma, x) \backsim N(\mu, \sigma^2)$. Let M be an apparatus measuring statistically relevant information about X and g. If a measurement M results in one single real number y then the following statement holds true:

if y bares information about the location of X with precision σ,

then y cannot bare more information about σ^{-1} than with precision $\sqrt{\frac{\pi-2}{2}}\sigma^{-1}$ and vice versa.

Interpretation remark: *This is true for every given quantized value of $\hbar > 0$, meaning that $\hbar$ drops out of the equations if applied to statistics.*

This is only one example for potential applications outside quantum physics. A more detailed discussion about the interpretation of the findings of this brief paper could lead to further applications, e.g. also in finance. Especially a discussion with regard to the interpretation remark of the above mentioned statistical application might be very fruitful. Might there be a kind of "generalization principle" of the uncertainty relations for normal distributions? Could we claim, e.g., that every normal distribution $g \backsim N(\mu, \sigma^2)$ can be regarded as generated by a coherent ground state of realizations of one quantized action $(\hbar =)$ ⓔ, which could have meaning also outside the realm of quantum physics?